J. DIFFERENTIAL GEOMETRY
26 (1987) 169–173

SPLIT RANK AND SEMISIMPLE AUTOMORPHISM GROUPS OF G-STRUCTURES

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1. Introduction

This paper is a continuation of the investigation begun in [1], [3], [4] concerning the semisimple automorphism groups of G-structures on compact manifolds. In those papers we were concerned with semisimple groups that preserve a structure which is algebraic and which defines a volume density, i.e. where the structure group G is an algebraic subgroup of $\mathrm{SL}'(n, \mathbb{R})$, the matrices with $|\det| = 1$. (For higher order structures we assumed that G is an algebraic subgroup of $\mathrm{SL}'(n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{R})^{(k)}$, the latter being the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ fixing the origin.) One of the basic conclusions in the above papers is that for any simple noncompact Lie group H preserving such a G-structure, we must have that H locally embeds in G. (In fact a stronger assertion is proven. See the above papers and Theorem 2 below.) The main goal of the present paper is to consider the situation in which H is no longer assumed to define a volume density. In this case natural examples easily show that one cannot expect a local embedding of H in G. However, our main result asserts that a basic structural invariant of H must be visible in G. More precisely, we prove:

Theorem 1. *Let H be a semisimple Lie group with finite center and suppose that H acts smoothly on a compact manifold M so as to preserve a G-structure on M, where G is a real algebraic group. Then* $\mathbb{R}\text{-rank}(H) \leqslant \mathbb{R}\text{-rank}(G)$.

We recall that the $\mathbb{R}$-rank, or split rank, of a real algebraic group is the maximal dimension of an algebraic torus that is diagonalizable over $\mathbb{R}$. For a semisimple Lie group H, $\mathrm{Ad}(H)$ will be the connected component of the identity of a real algebraic group, and the $\mathbb{R}$-rank, or split rank, of H is defined to be the split rank of this real algebraic group. We shall also clear up

Received June 13, 1986. The author's research was partially supported by a National Science Foundation Grant.

a point that was left open in [1], [3] concerning the case in which G defines a volume density. Namely, the general results in [3] were established for noncompact simple groups, not for semisimple groups. In [3], a special argument was given that clarified the semisimple situation for the case of Lorentz structures. Here we observe that a simple argument enables us to extend the results of [3] to the semisimple case in general, at least in the case of finite center.

Theorem 2 (*cf.* [1], [3], [4]). *Let H be a connected semisimple Lie group with finite center and no compact factors, and suppose that H acts on a compact n-manifold preserving a G-structure, where G is algebraic and defines a volume density. Then there is an embedding of Lie algebras $\mathfrak{h} \to \mathfrak{g}$. Furthermore, the representation $\mathfrak{h} \to \mathfrak{g} \to \mathfrak{sl}(n, \mathbb{R})$ contains $\mathrm{ad}_{\mathfrak{h}}$ as a direct summand.*

Part of this work was done while the author was a visitor at Harvard University. We would like to thank the members of that department for their hospitality.

2. Preliminaries

We establish here some preliminary information we shall need for the proofs of Theorems 1 and 2.

Proposition 3. *Let H be a connected semisimple Lie group with finite center, acting smoothly on a connected manifold M, and assume $p \in M$ is a fixed point. Let π: $H \to \mathrm{GL}(TM_p)$ be the corresponding representation at p. If π is trivial, then H acts trivially on M.*

Proof. Let $K \subset H$ be a maximal compact subgroup. It suffices to see that K acts trivially. For a compact group, any smooth action can be linearized around fixed points, so the set of invariant frames for the tangent bundle is both open and closed.

Proposition 4. *Suppose H is a connected semisimple Lie group with finite center, acting smoothly on a connected manifold M. If the set of fixed points has positive measure, then H acts trivially.*

Proof. If the set of fixed points, F, has positive measure, choose a density point p for F in the sense of Lebesgue. Then any small ball around p intersects F in a set of positive measure. The action of the maximal compact subgroup $K \subset H$ can be linearized around p, which implies that K leaves a set of vectors in TM_p invariant which has positive measure in TM_p. It follows that this linear representation of K is trivial, and the proof of Proposition 3 completes the proof.

If a Lie group H acts smoothly on a manifold M, and $m \in M$, we let H_m be the stabilizer of m in H, and $\mathfrak{h}_m \subset \mathfrak{h}$ the Lie algebra of H_m. If V is a vector space we let $\mathrm{Gr}_d V)$ be the Grassman variety of d-dimensional linear subspaces.

For a Lie group L, we let L^0 be the identity component. If a Lie group L is the identity component of an algebraically connected real algebraic group, by a rational homomorphism of L into a real algebraic group we mean the restriction of (a necessarily unique) rational homomorphism of the ambient algebraic group. The following is standard.

Lemma 5. *Suppose H is a Lie group acting smoothly on a manifold M. Let d be the minimal dimension of an H-orbit in M. Then $M_1 = \{m \in M \mid \dim(Gm) = d\}$ is closed, and the map $m \to \mathfrak{h}_m$ defines a continuous map $\varphi\colon M_1 \to \mathrm{Gr}_q(\mathfrak{h})$, where $q = \dim(H) - d$. Further, φ is an H-map, where H acts on $\mathrm{Gr}_q(\mathfrak{h})$ via* $\mathrm{Ad}(H)$.

We recall briefly the notion of the algebraic hull of a cocycle defined for an ergodic group action (see [4] or [2] for an elaboration). Suppose that H is a locally compact group acting ergodically on a standard measure space (M, μ). Suppose that G is a real algebraic group and that $\alpha\colon H \times M \to G$ is a cocycle, i.e., the following identity is satisfied (for each $h_1, h_2 \in H$, and almost all $m \in M$): $\alpha(h_1 h_2, m) = \alpha(h_1, h_2 m)\alpha(h_2, m)$. We recall that two cocycles α, β are called equivalent if there is a measurable $\varphi\colon M \to G$ such that for each h and almost all m, $\beta(h, m) = \varphi(hm)^{-1}\alpha(h, m)\varphi(m)$.

Lemma 6 ([2], [4], [5]). *There is an algebraic subgroup $L \subset G$ with the following properties*:

(i) *α is equivalent to a cocycle taking all its values in L.*

(ii) *For any proper algebraic subgroup $L' \subset L$, α is not equivalent to a cocycle taking all its values in L'.*

(iii) *Up to conjugacy in G, L is the unique algebraic subgroup satisfying* (i), (ii).

(iv) *If α is equivalent to a cocycle taking all its values in some closed subgroup $L_0 \subset G$, then some conjugate of L_0 is contained in L.*

L is then called the algebraic hull of α, and it is well defined up to conjugacy in G. The following property is easily established.

Lemma 7. *Suppose $p\colon G_1 \to G_2$ is a rational homomorphism of real algebraic groups. If α is a G_1-valued cocycle with algebraic hull L_1, then the algebraic hull of the G_2-valued cocycle $p \circ \alpha$ is the algebraic hull of $p(L_1)$ (in which, we recall, $p(L_1)$ is a subgroup of finite index).*

3. Proof of Theorem 1

Let M_1 be as in Lemma 5. Since M_1 is a compact H-space, we can choose a minimal H-space $M_0 \subset M_1$, i.e., a closed H-invariant subset in which every orbit is dense. Then, letting φ be as in Lemma 5 as well, we have that $\varphi(M_0) \subset \mathrm{Gr}_q(\mathfrak{h})$ is minimal. However, the action of H on $\mathrm{Gr}_q(\mathfrak{h})$ is algebraic, and hence every orbit is locally closed. It follows that $\varphi(M_0)$ consists of a single compact H-orbit. Fix $x \in M_0$, and let $\mathfrak{h}_x = \mathfrak{h}_0$. Then we can consider φ as an H-map $\varphi\colon M_0 \to \mathrm{Ad}(H)/N(\mathfrak{h}_0) \subset \mathrm{Gr}_q(\mathfrak{h})$, where $N(\mathfrak{h}_0)$ is the normalizer of $\mathfrak{h}_0$ in $\mathrm{Ad}(H)$. In particular, the algebraic subgroup $N(\mathfrak{h}_0)$ is cocompact in $\mathrm{Ad}(H)$, and therefore we can find a maximal $\mathbb{R}$-split torus T of $\mathrm{Ad}(H)$, with $T \subset N(\mathfrak{h}_0)$.

Let $\mathfrak{n}$ be the Lie algebra of $N(\mathfrak{h}_0)$, so that $\mathfrak{h}_0 \subset \mathfrak{n}$ is an ideal. The adjoint representation yields a rational (and in particular semisimple) representation $T \to \mathrm{GL}(\mathfrak{h}/\mathfrak{h}_0)$. Let $T_0 \subset T$ be the kernel, so that T_0 is an $\mathbb{R}$-split subtorus. Since the representation of T_0 on $\mathfrak{h}$ is semisimple, we can write $\mathfrak{h} = \mathfrak{h}_0 \oplus W$, where $W \subset \mathfrak{h}$ is a subspace and T_0 acts trivially on W. In particular, $\mathfrak{h}_0$ contains all the root spaces of T_0 acting via Ad_H on $\mathfrak{h}$ corresponding to nontrivial roots. The algebra generated by the nontrivial root spaces for T_0 is an ideal, and hence $\mathfrak{h}_0$ contains an ideal of $\mathfrak{h}$, say $\mathfrak{h}_1$, containing all the nontrivial root spaces for T_0. Thus we can write $\mathfrak{h}$ as a sum of ideals, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where T_0 acts trivially on $\mathfrak{h}_2$. Since $\mathfrak{h}_2$ is semisimple, it follows that $\mathfrak{t}_0 \subset \mathfrak{h}_1 \subset \mathfrak{h}_0$. Let H_1 be the connected normal subgroup of H corresponding to $\mathfrak{h}_1$. Then $\mathfrak{h}_1$, and hence H_1, acts trivially on $\mathfrak{h}/\mathfrak{h}_0$, and by Proposition 3, H_1 acts locally faithfully on $T(M)_x/T(Hx)_x$. In particular, T_0 acts rationally and locally faithfully on $T(M)_x/T(Hx)_x$. Let T_1 be a split torus complementary to T_0 in T. We then have that $T = T_0 \times T_1$, and T_1 acts faithfully on $\mathfrak{h}/\mathfrak{h}_0$.

Now let $M_2 \subset M_0$ be a minimal $N(\mathfrak{h}_0)^0$ space. Since $(H_x)^0$ is normal in $N(\mathfrak{h}_0)$, it fixes all points of $N(\mathfrak{h}_0)x$, and hence fixes all points in the closure of this orbit, in particular all points in M_2. Since the dimension of all stabilizers in H of points in M_0 are the same, we have $\mathfrak{h}_m = \mathfrak{h}_0$ for all $m \in M_2$. Thus for $m \in M_2$, we can identify the tangent space to the H-orbit through m with $\mathfrak{h}/\mathfrak{h}_0$. The representation of H_1 on $T(M)_m/T(Hm)_m$ will vary continuously over $m \in M_2$, and since H_1 is semisimple and M_2 is connected, all these representations are equivalent. In particular, the representations of $(T_0)^0$ on these spaces are all rational, and all equivalent.

Choose a probability measure on M_2 which is invariant and ergodic under T^0 [2, Chapter 4]. Let $\alpha\colon T^0 \times M_2 \to \mathrm{GL}(n, \mathbb{R})$ be a cocycle corresponding to the action of T^0 on the tangent bundle of M over the space M_2 (cf. [4]). Let L be the algebraic hull of this cocycle. Since H, and in particular T^0, leaves a G-structure on M invariant, we have (up to conjugation) $L \subset G$. By our observations above, we can measurably trivialize TM over M_2 in such a way that $TM \cong M \times \mathbb{R}^n$, $\mathbb{R}^n = V_1 \oplus V_2$, $V_1 \cong \mathfrak{h}/\mathfrak{h}_0$, such that for $t \in T_0^0$, we have

$$\alpha(m, t) = \begin{pmatrix} I & 0 \\ 0 & \pi_2(t) \end{pmatrix},$$

where π_2 is a faithful rational representation, and for $t \in T_1^0$, we have

$$\alpha(m,t) = \begin{pmatrix} \pi_1(t) & * \\ 0 & * \end{pmatrix},$$

where $\pi_1(t)$ is Ad(t) acting on $\mathfrak{h}/\mathfrak{h}_0$, and, as we remarked above, is a faithful rational representation. Let β be the projection of α in $\mathrm{GL}(V_1) \times \mathrm{GL}(V/V_1)$. To prove the theorem, it suffices to see that the split rank of L is at least as large as $\dim(T)$, and by Lemma 7, to prove this it suffices to see that the split rank of the algebraic hull of β is at least $\dim(T)$. Thus, we need only see that if π is a faithful rational representation of T^0, then the algebraic hull of the cocyle $\beta(m,t) = \pi(t)$ $(m \in M_2)$ is locally isomorphic to T. Let T^* be the algebraic hull of the group $\pi(T^0)$; then T^* is a split torus, $\pi(T^0) \subset T^*$ is of finite index, and $\dim(T) = \dim(T^*)$. If β is equivalent to a cocycle into $Q \subset T^*$, then there is a measurable T^0-map φ: $M_2 \to T^*/Q$. Since there is a finite T^0-invariant measure on M_2, there is one on T^*/Q as well, and if Q is algebraic, it is clear that $\dim(Q) = \dim(T)$. This complete the proof.

4. Proof of Theorem 2

The argument of [1, Lemma 6], using the Borel density theorem, shows that the Lie algebra of the stabilizer of almost every point is an ideal. By Proposition 4, it follows that almost every point has a discrete stabilizer. The proof then follows as in the simple case, as in [3] or [4].

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